2. Ü 20.11.20

$$a_n = 1 + 2^{2^n} + 2^{2^{n+1}}, \qquad n \geq 0$$

$$a_0 = 1 + 2^1 + 2^2 = 7 \quad \checkmark$$

$$a_n^2 = \left(1 + 2^{2^n} + 2^{2^{n+1}}\right)^2$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$$

$$= 1 + 2^{2\cdot 2^n} + 2^{2\cdot 2^{n+1}} + 2\left(2^{2^n} + 2^{2^{n+1}} + 2^{2^n + 2^{n+1}}\right)$$

$$= \underbrace{1 + 2^{2^{n+1}} + 2^{2^{n+2}}} + 2\left(\quad\right)$$

$$= a_{n+1} + 2 \cdot 2^{2^n}\left(1 + \underline{2^{2^n} + 2^{2^{n+1}}}\right)$$

$$= a_{n+1} + 2^{2^n+1} \cdot a_n$$

Also:

$$a_{n+1} = a_n^2 - 2^{2^n+1} \cdot a_n$$

2. a. $\displaystyle\prod_{k=1}^{n} (1+x_k) \geq 1 + \sum_{k=1}^{n} x_k$

for $0 < x_k < 1$.

$n=1$: $\quad 1+x_1 \geq 1+x_1$ ✓

$n \Rightarrow n+1$:

b. $$\prod_{k=1}^{n+1} (1+x_k) = \underbrace{\left(\prod_{k=1}^{n} (1+x_k)\right)}_{IA} \cdot (1+x_{n+1})$$

$$\geq \left(1 + \sum_{k=1}^{n} x_k\right) \cdot (1+x_{n+1})$$

$$= 1 + \sum_{k=1}^{n} x_k + x_{n+1} + x_{n+1} \cdot \sum_{k=1}^{n} x_k$$

$$= 1 + \sum_{k=1}^{n+1} x_k + \underbrace{x_{n+1} \cdot \sum_{k=1}^{n} x_k}_{\geq 0}$$

$$\geq 1 + \sum_{k=1}^{n+1} x_k .$$

c. $n=1$: $\quad x_1 \cdot \frac{1}{x_1} = 1 \geq \frac{1}{n^2}\Big|_{n=1}$ ✓

$n \Rightarrow n+1$:

$$\left(\sum_{k=1}^{n+1} x_k\right) \cdot \left(\sum_{k=1}^{n+1} \frac{1}{x_k}\right)$$

$$= \left(x_{n+1} + \sum_{k=1}^{n} x_k\right)\left(\frac{1}{x_{n+1}} + \sum_{k=1}^{n} \frac{1}{x_k}\right)$$

$$= \underbrace{\sum_{k=1}^{n} x_k \cdot \sum_{k=1}^{n} \frac{1}{x_k}} + x_{n+1} \sum_{k=1}^{n} \frac{1}{x_k} + \frac{1}{x_{n+1}} \cdot \sum_{k=1}^{n} x_k + 1$$

$$\geq n^2 + \underbrace{\sum_{k=1}^{n} \Big(\underbrace{\frac{x_{n+1}}{x_k} + \frac{x_k}{x_{n+1}}}_{a + \frac{1}{a} \geq 2}\Big)}_{\geq 2n} + 1$$

$$\geq n^2 + 2n + 1$$

$$= (n+1)^2 \qquad \blacksquare$$

$0, 1, 1, 2, 3, 5, 8, \ldots$

$$\begin{cases} f_{n+1} = f_n + f_{n-1}, & n \geq 1 \\ f_0 = 0 \\ f_1 = 1 \end{cases}$$

<u>Ansatz</u>: $f_n = a\lambda^n + b\mu^n, \quad n \geq 0$

a. $0 = f_0 = a + b$

$1 = f_1 = a\lambda + \mu b$

Dann

$(\mu - \lambda)\, a = -1 \qquad a = \dfrac{1}{\lambda - \mu}$

$(\mu - \lambda)\, b = 1 \qquad b = \dfrac{1}{\mu - \lambda}$

Also:

$$\boxed{f_n = \frac{\lambda^n - \mu^n}{\lambda - \mu}}.$$

Damit

$$1 = f_2 = \frac{\lambda^2 - \mu^2}{\lambda - \mu} = \lambda + \mu$$

b. $f_{n+1} = f_n + f_{n-1}$

Also

$$\boxed{\lim \frac{f_{n+1}}{f_n} = \frac{f_n + f_{n-1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} = 1 + \frac{1}{\lim \frac{f_n}{f_{n-1}}}}$$

$\lambda = 1 + \frac{1}{\lambda}$

Dabei

$$\frac{f_n}{f_{n-1}} = \frac{\lambda^n - \mu^n}{\lambda^{n-1} - \mu^{n-1}}$$

$$= \frac{\lambda^n}{\lambda^{n-1}} \cdot \frac{1 - \left(\frac{\mu}{\lambda}\right)^n}{1 - \left(\frac{\mu}{\lambda}\right)^{n-1}}$$

$$= \lambda \cdot \frac{1 - x^n}{1 - x^{n-1}} \to 1, \quad x = \frac{\mu}{\lambda}$$

① Bzw. $|\lambda| > |\mu|$, also $|x| < 1$

$|x|^n \longrightarrow 0, \quad n \to \infty$

$$\boxed{\lim_{n\to\infty} \frac{f_n}{f_{n-1}} = \lambda}$$

und $\lambda = 1 + \frac{1}{\lambda}$

λ is big ~~so~~ being $\lambda^2 = \lambda+1$

$\mu = 1-\lambda$ is smaller big so

$\mu^2 = \mu+1$. (i)

$\lambda^2 - \lambda + 1$.

$$\lambda = \frac{1+\sqrt{5}}{2}, \quad \mu = \frac{1-\sqrt{5}}{2}$$

c. Binet's Formula:

$$f_n = \frac{\lambda^n - \mu^n}{\lambda - \mu}, \qquad \lambda^2 = \lambda+1, \quad \mu^2 = \mu+1$$

$n = 0, 1, 2$ ✓

$n \Rightarrow n+1$:

$$\begin{aligned}
(\lambda-\mu) f_{n+1} &= (\lambda-\mu)(f_n + f_{n-1}) \\
&= \lambda^n - \mu^n + \lambda^{n-1} - \mu^{n-1} \\
&= \lambda^{n-1}(\lambda+1) - \mu^{n-1}(\mu+1) \\
&= \lambda^{n-1}\cdot\lambda^2 - \mu^{n-1}\cdot\mu^2 \\
&= \lambda^{n+1} - \mu^{n+1}.
\end{aligned}$$

∎

$$\sum_{k=1}^{n} \binom{n}{k} f_k = f_{2n}$$

$$(\lambda-\mu) \sum_{k=1}^{n} \binom{n}{k} f_k = \sum_{k=1}^{n} \binom{n}{k} (\lambda^k - \mu^k)$$

$$= \sum_{k=1}^{n} \binom{n}{k} \lambda^k - \sum_{k=1}^{n} \binom{n}{k} \mu^k$$

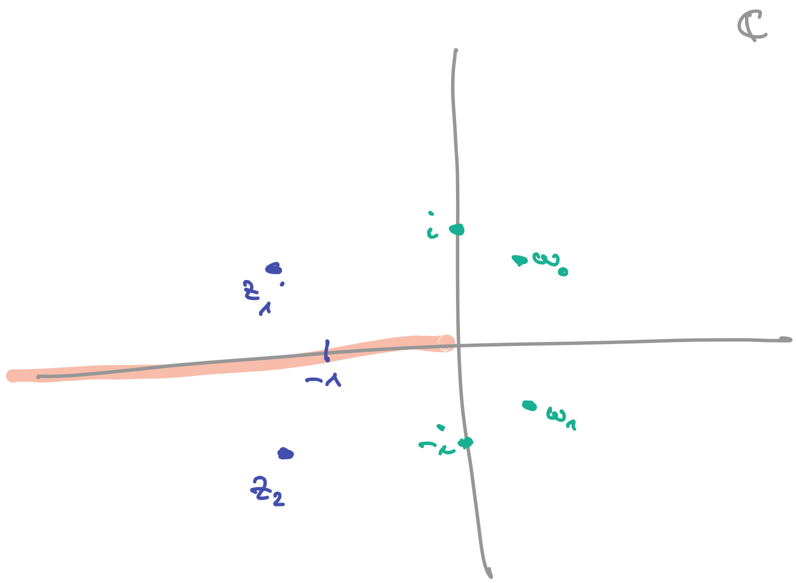
$k=0$: $\quad 1 \qquad 1$

$$= \sum_{k=0}^{n} \binom{n}{k} \lambda^k - \sum_{k=0}^{n} \binom{n}{k} \mu^k$$

$$= (1+\lambda)^n - (1+\mu)^n$$

$$= \lambda^{2n} - \mu^{2n}.$$

$z \in (-\infty, 0]$



$$z = x + iy \quad , \quad w = u + iv$$

$$w^2 = z$$

Also:

$$w^2 = u^2 - v^2 + 2iuv = x + iy = z$$

Also

$$\begin{cases} u^2 - v^2 = x \\ 2uv = y \end{cases}$$

Wegen $u = \operatorname{Re} w > 0$. Dann

$$v = \frac{y}{2u}$$

Also:

$$u^2 - \frac{y^2}{4u^2} = x$$

$$\Longleftrightarrow \quad \boxed{u^4 - u^2 x - \frac{y^2}{4} = 0}$$

Quadratische Gleichung für u^2:

Also:

$$u^2 = \frac{x}{2} \pm \sqrt{\frac{x^2}{4} + \frac{y^2}{4}}$$

$$= \frac{x}{2} \pm \frac{1}{2}\sqrt{|z|^2}$$

$$= \frac{x}{2} \pm \frac{1}{2}|z| \geq 0$$

Also:

$$\boxed{u^2 = \frac{1}{2}\left(|z| + x\right)}$$

Also:

$$u = \sqrt{\frac{|z| + x}{2}} \geq 0$$

Und $\quad u^2 + v^2 = |w|^2 = |z| \quad$ Somit:

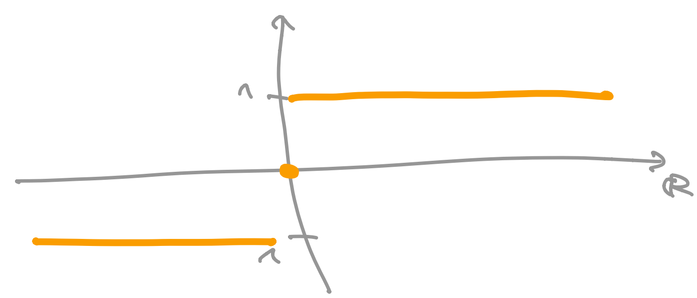
$$\boxed{v^2 = |z| - u^2 = \frac{|z| - x}{2}} \geq 0$$

Vorzeichen von v:

$$2\underbrace{u}_{\geq 0}v = y$$

Also

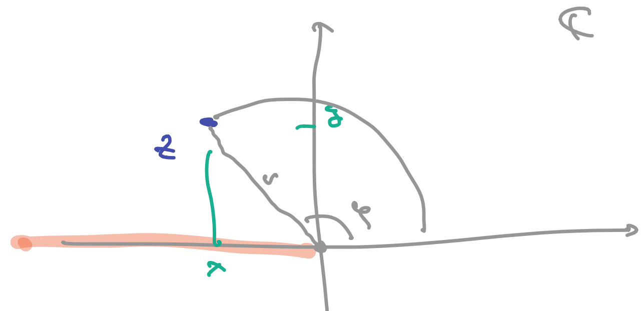
$$\operatorname{sgn}(v) = \operatorname{sgn}(y)$$



$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Dann jetzt auch algebraisch:

Polardarstellung $\boxed{z = re^{i\varphi}}$



Wähle:

$$w = s e^{i\psi} \quad \text{mit} \quad w^2 = s^2 e^{2i\psi} = z = re^{i\varphi}$$

Also:

$$w = \underbrace{\sqrt{r}}_{\text{reell}}\, e^{i\varphi/2}$$

eindeutig für $-\pi < \varphi < \pi$.