22. Vorlesung 3. 2. 2021

$$\exp' z = \sum_{n\geq 0} \left(\frac{z^n}{n!}\right)'$$

$$= \sum_{n\geq 0} \frac{n}{n!} z^{n-1}$$

$$= \sum_{n\geq 1} \frac{z^{n-1}}{(n-1)!} \quad = \sum_{n\geq 0} \frac{z^n}{n!}$$

$$= \exp z .$$

$$\sin' z = \cos z ,$$

$$\cos' z = -\sin z .$$

Additionstheorem :

$$\exp(z+w) = \exp(z) \cdot \exp(w).$$

Dem.:

$$\exp(iz) = \sum_{n \geq 0} \frac{(iz)^n}{n!}$$

$$= \sum_{n \geq 0} (-1)^n \frac{z^{2n}}{(2n)!} + i \sum_{n \geq 0} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$= \sin(z) + i \cdot \cos(z). \quad \blacksquare$$

Für reelle t:

$$e^{it} := \exp(it) = \cos t + i \sin t$$

$t = \pi$:

$$e^{i\pi} = -1$$

$$\Longleftrightarrow \quad \boxed{e^{i\pi} + 1 = 0.}$$

$z^{cf} \in \mathbb{C} \sim (\text{conf}, \text{dist}) \in \mathbb{R}^2$

$\mathbb{C}$

$z^{cf} = \text{conf} + i\,\text{dist}$

$= (\text{conf}, \text{dist}) \quad \mathbb{R}^2$

$$e^{it} = \cos t + i\sin t$$

$$e^{i(s+t)} = \underbrace{\cos(s+t)}_{} + i\,\underbrace{\sin(s+t)}_{}$$

$$\Rightarrow \quad e^{is}e^{it} = (\cos s + i\sin s)(\cos t + i\sin t)$$

$$= \underbrace{\cos s\cdot\cos t - \sin s\,\sin t}_{} + i\,\underbrace{(\sin s\,\cos t + \cos s\,\sin t)}_{}$$

Also:

$$\cos(s+t) = \cos s\cdot\cos t - \sin s\cdot\sin t$$

$$\ldots$$

$$\underbrace{(\cos t + i\sin t)^n}_{} = (e^{it})^n = e^{int} = \underbrace{\cos(nt)}_{} + i\,\underbrace{\sin(nt)}_{}$$

$$\begin{cases} e^{it} = \cos t + i\sin t \\ e^{-it} = \cos t - i\sin t \end{cases}$$

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \mathrm{Re}\, e^{it}$$

$$\sin t = \frac{e^{it} - e^{-it}}{2i} = \mathrm{Im}\, e^{it}.$$

$z \neq 0$: $\quad z = |z| \cdot \vartheta$,

$$\vartheta = \frac{z}{|z|} \in S \quad : \quad |\vartheta| = 1$$

Dann $\quad \vartheta = e^{i\varphi}, \quad \varphi \in [0, 2\pi)$

Also:

$$z = |z| \cdot \vartheta = |z| e^{i\varphi}$$

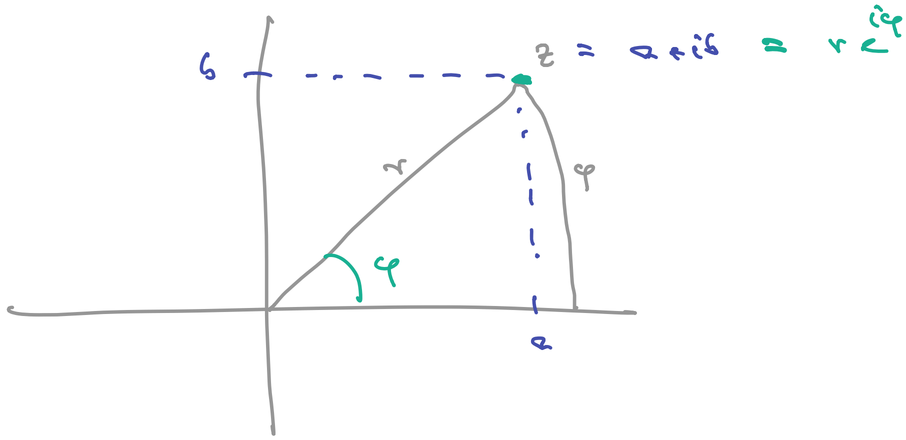
$$= r e^{i\varphi}, \quad r = |z| > 0.$$

Conjubel:

$$|z| = |r| \cdot \underbrace{|e^{i\varphi}|}_{1} = r,$$

$$z = r e^{i\varphi} = r e^{i(\varphi + 2\pi n)}, \quad n \in \mathbb{Z}$$

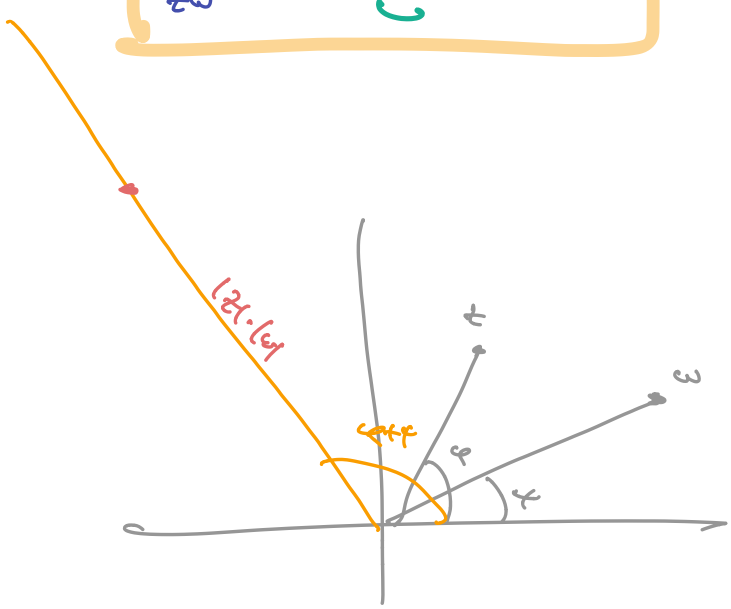
$$= r e^{i\varphi} \cdot \underbrace{e^{2\pi n i}}_{1}.$$



Bem.:

$z = re^{i\varphi}$, $w = se^{i\psi}$:

$$\boxed{zw = rs\,e^{i(\varphi+\psi)}}$$



Bew: $z=0$ keine Einheitswurzel.

Ansatz: $z = re^{i\varphi}$

Dann: $z^n = r^n e^{in\varphi} \overset{!}{=} 1$

Dann: $r = 1$ ✓

Jetzt: $e^{in\varphi} = 1$

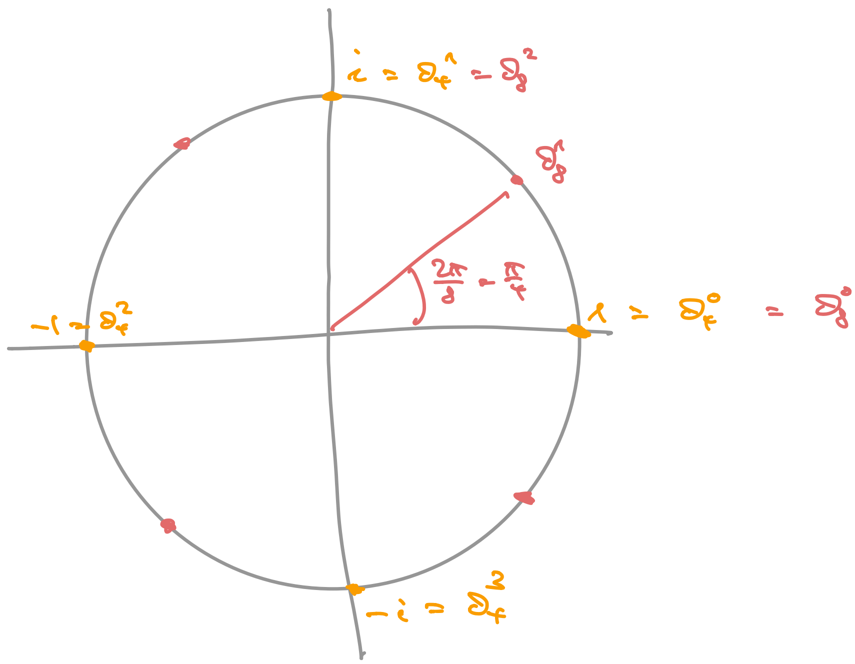
$n\varphi = 2\pi k \,, \quad k \in \mathbb{Z}$

$\iff \varphi = 2\pi \frac{k}{n}$.

$z = e^{2\pi i \cdot \frac{k}{n}} \,, \quad k = \underbrace{0, 1, \dots, n-1}_{n}$.

$= e^{(2\pi i \frac{1}{n})k}$

$= \Big(\underbrace{e^{2\pi i \frac{1}{n}}}_{\zeta_n}\Big)^k = \zeta_n$



Dem: $\quad z = r e^{i\varphi} \neq 0$.

$$\omega_0 = r^{1/n} e^{i\varphi/n}:$$

$$\omega_0^n = r\left(e^{i\varphi/n}\right)^n = r e^{i\varphi} = z.$$

Dann

$\frac{\omega}{\omega_0}$ ist Einheitswurzel:

$$\left(\frac{\omega}{\omega_0}\right)^n = \frac{\omega^n}{\omega_0^n} = 1.$$

$$\frac{\omega}{\omega_0} = \zeta_n^k \implies \omega = \omega_0 \zeta_n^k$$

$\varphi' = \varphi \,, \qquad \varphi(0) = 1$

$\varphi'' = -\varphi \,, \qquad \boxed{\varphi(0) = 0, \quad \varphi'(0) = 1}$

...

$\varphi'' = \varphi \,, \qquad$ ——— " ———

$e^t \,, \; e^{-t}$ Basis Lösung, also auch

$a e^t + b e^{-t}.$

$$\begin{cases} a + b = 0 \\ a - b = 1 \end{cases} \qquad \begin{matrix} a = \frac{1}{2} \\ b = -\frac{1}{2} \end{matrix} \quad : \quad \frac{e^t - e^{-t}}{2}$$

$$\sinh t = \sum_{n \geq 0} \frac{t^{2n+1}}{(2n+1)!} = \frac{e^t - e^{-t}}{2}$$

$$\cosh t = \sum \frac{t^{2n}}{(2n)!} = \frac{e^t + e^{-t}}{2}.$$

$$\cosh^2 t - \sinh^2 t$$

$$= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4}$$

$$= \frac{2}{4} + \frac{2}{4}$$

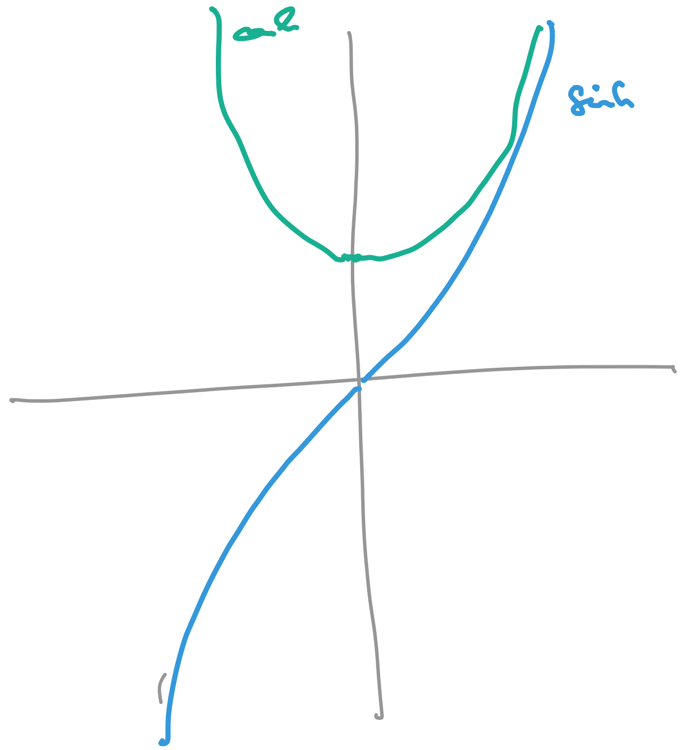
$$= 1.$$

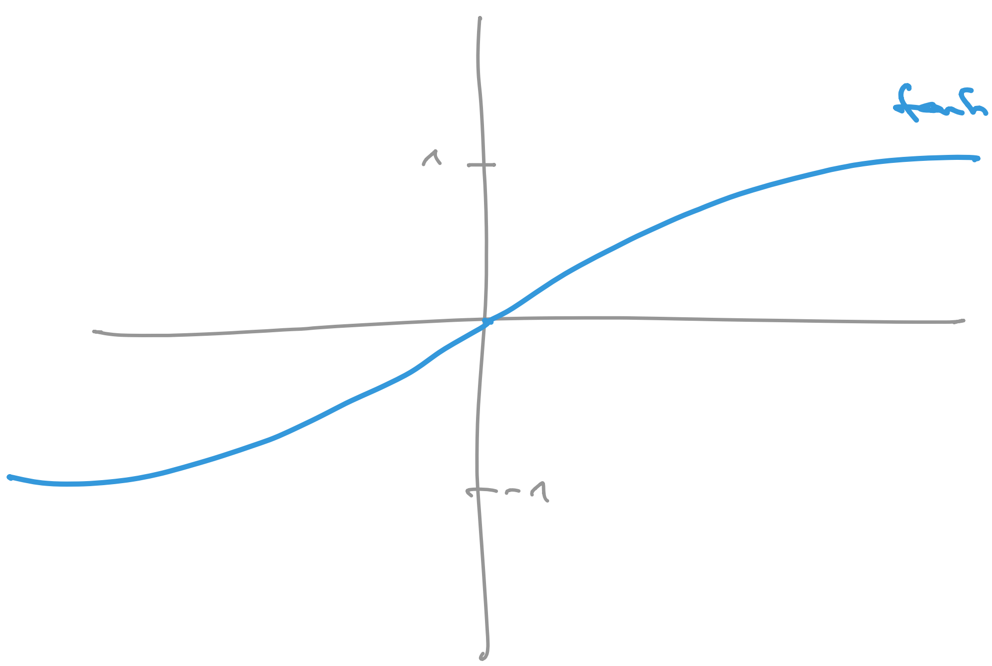
$$\underline{\sin(it)} = \frac{e^{it} - e^{-it}}{2i}\Big|_{z=it}$$

$$= \frac{1}{2i}\left(e^{-t} - e^{t}\right)$$

$$= \frac{1}{i}\,\frac{e^{t} - e^{-t}}{2}$$

$$= \underline{-i\,\sinh t}$$


cosh
sinh


tanh
1
-1

Betrachte :

$$t = \cosh s = \frac{e^{s} + e^{-s}}{2} \geq 1$$

$$\iff \quad 2t = e^{s} + e^{-s}$$

$$\iff \quad e^{2s} + 1 = 2t\, e^{s}$$

$$\iff \quad \boxed{e^{2s} - 2t\, e^{s} + 1 = 0} \qquad e^{s}$$

$$e^{s} = t \pm \sqrt{t^2 - 1} \qquad \geq 1$$

für $+$

$$e^{s} = t + \sqrt{t^2-1}$$

$$s = \log\left(t + \sqrt{t^2-1}\right).$$