21. Vorlesung 23. 1. 2021

(i) $\sin(-t) = -\sin(t)$ ungerade F.

$\cos(-t) = \cos(t)$ gerade F.

(ii) $\sin'(t) = \cos(t)$ folgt direkt

$\cos'(t) = -\sin(t)$ durch Diff

der PR.

(iii) $\sin^2(t) + \cos^2(t) = 1$.

$s = \sin$, $c = \cos$:

$s^2 + c^2 \equiv 1$:

Bew:

$$(s^2+c^2)' = 2ss' + 2cc'$$
$$= 2sc - 2sc$$
$$= 0 \quad \text{für alle } t$$

Also:

$$s^2 + c^2 \equiv \text{const}$$
$$= \cancel{s^2(0)} + \underbrace{c^2(0)}_{1}$$
$$= 1.$$

$$\sin(s+t) = \sin(s)\cos(t) + \cos(s)\sin(t) :$$

Beide sind Lösungen von $\varphi'' = -\varphi$ (als Fkt. in t)

AW bei $t=0$:

$$\sin(s)\underbrace{\cos(0)}_{1} + \cos(s)\underbrace{\sin(0)}_{0}$$

$$= \sin(s)$$

$$= \sin(s+t)\big|_{t=0} .$$

Die Zahl π.

$\pi > 0$:

$$\sin(\pi) = 0 ,$$

$$\sin(t) \underset{>}{\neq} 0 , \quad 0 < t < \pi .$$

Beweis: $s = \sin$, $c = \cos$.

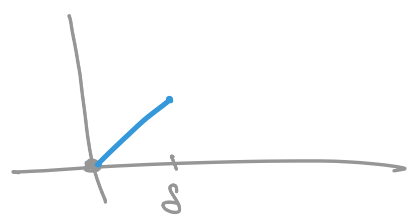
$$c(0) = 1$$

$\Longrightarrow$ $c(t) > 0$, $0 \leq t \leq \delta$

$\Longrightarrow$ $s' = c > 0$, auf $[0, \delta]$

$s(0) = 0$, s steig $\nearrow$

$\Longrightarrow$ $\boxed{s(t) > 0 \,, \quad 0 \leq t \leq \delta}$



Angenommen, es gilt

$$s \geq 0 \quad \text{auf } [0, \infty).$$

Dann:

$$c' = -s < 0 \quad \text{auf } [0, \infty)$$

$\Rightarrow$ c streng monoton fallend.

Zwei Möglichkeiten:

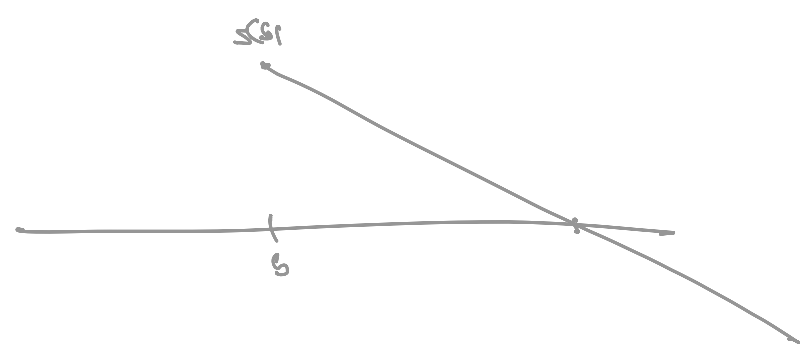
(i) c hat eine NS: $c(a) = 0$

(ii) c hat keine NS.

Zu (i): Dann gibt es ein b mit $c(b) < 0$

$$s'(t) = c(t) \leq c(b) < 0, \quad t \geq b$$

Dann hat die Lösung s (rechts von b) eine NS, also:



(ii) Free c Remain NS for $(c \neq 0)$:

Dem:

$$s' = c > 0$$

so

s stays nonneg strictly,

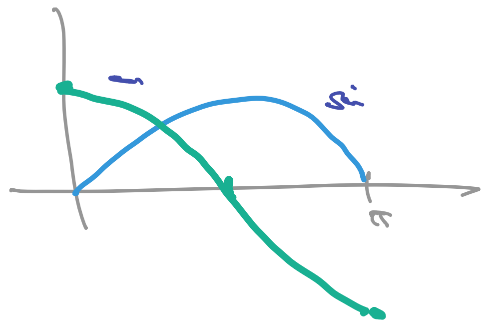
so

$$c'(t) = -s(t) \le -s(\delta) < 0, \quad t \ge \delta.$$

On the Fixed so c stays eventually Fixed,

with Rate $\le -s(\delta) < 0$. Also ...

c remain NS Strictly. $\blacksquare$



Beweis Satz 12:

$$c'(t) = -s(t) < 0, \quad 0 < t < \pi$$

Also c streng monoton fallend auf $[0, \pi]$.

Weiter:
$$c^2 + s^2 = 1,$$
$$s(\pi) = 0$$
$$\Rightarrow \quad c^2(\pi) = 1$$

Da $c(0) = 1$, folgt $c(\pi) = -1$.

Alt. Beweis:

$$2 \underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{>0} \underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0} = \sin(\pi) = 0$$

$$\Rightarrow \quad \sin^2\left(\tfrac{\pi}{2}\right) = 1 \quad \Rightarrow \quad \sin\left(\tfrac{\pi}{2}\right) = 1. \quad \blacksquare$$

$$s(t+\pi) = s(t)\,\overset{-1}{c(\pi)} + c(t)\,\overset{0}{s(\pi)}$$

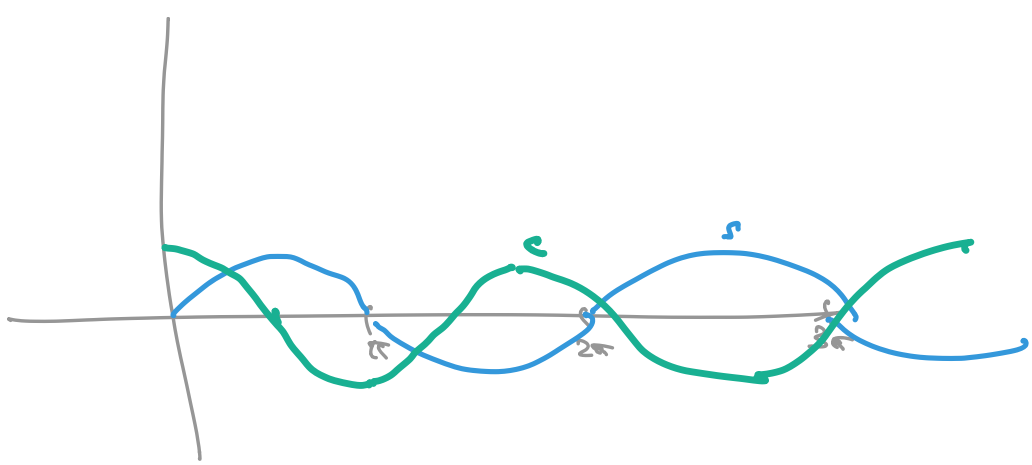
$$= -s(t)$$

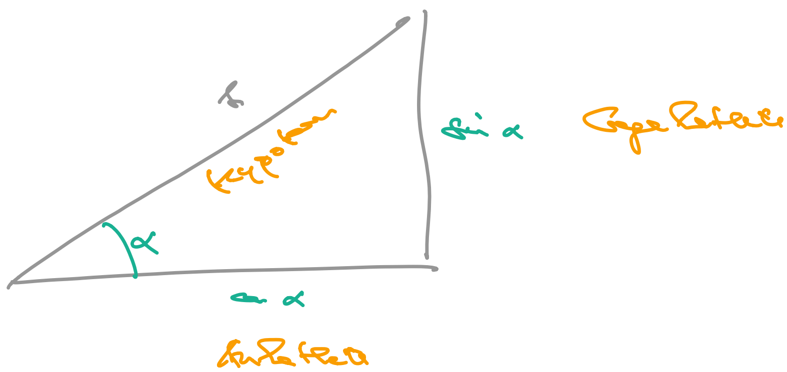
$$s(t+2\pi) = -s(t+\pi) = s(t).$$

$$c(t+\pi) = c(t)\,c(\pi) - s(t)\,s(\pi)$$

$$= -c(t)$$

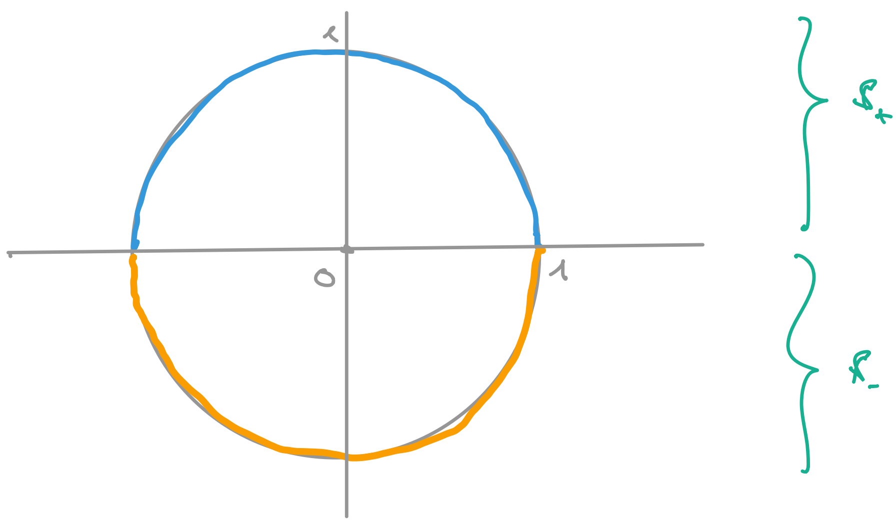
$$c(t+2\pi) = c(t).$$





Daten Bsp:

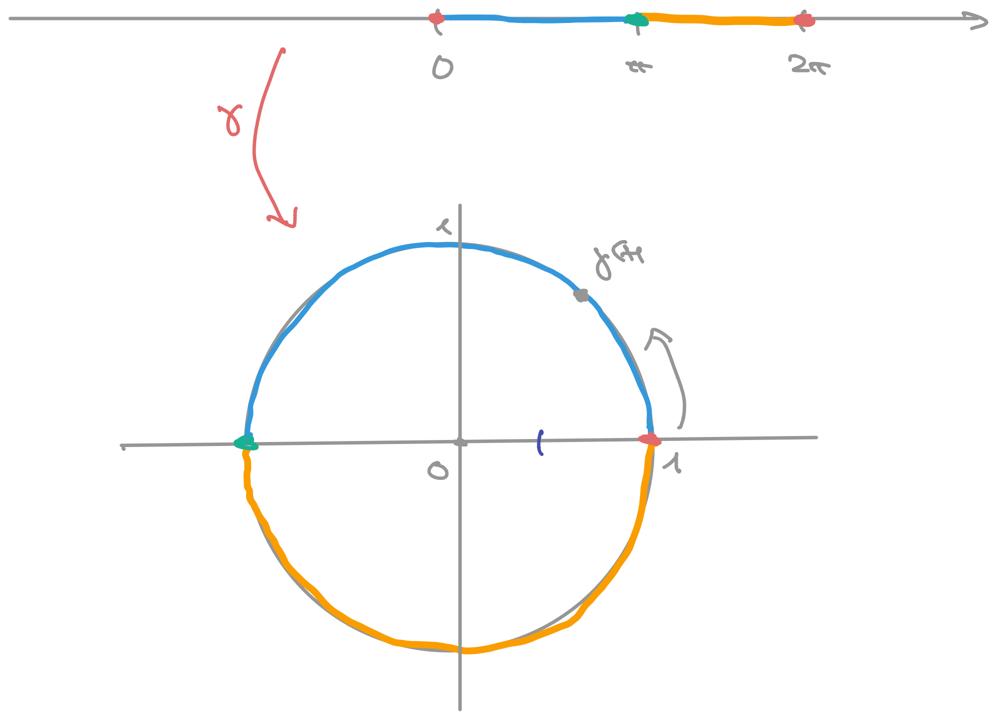
$$S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$



$$\gamma : \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto \gamma(t) = (\cos t, \sin t).$$

$$\cos^2(t) + \sin^2(t) = 1,$$

$$\Rightarrow \quad \gamma(t) \in S$$



Beweis: $\gamma : [0,\pi] \longrightarrow S_+$

und $\quad s(t) \geq 0 \quad$ auf $[0,\pi]$

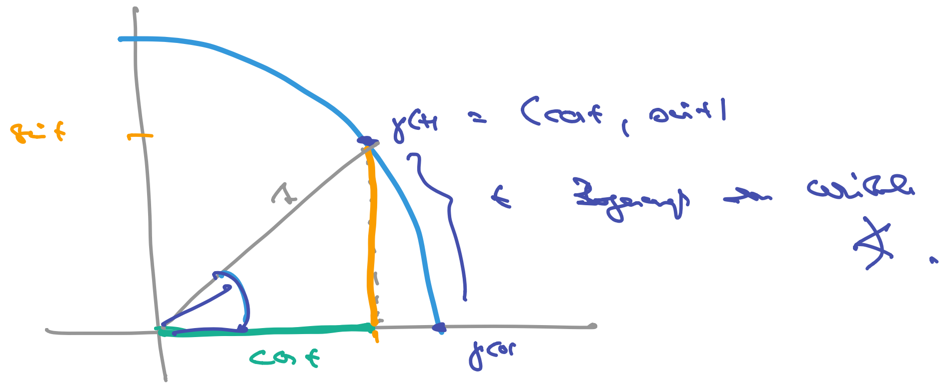
Da cos auf $[0,\pi]$ streng monoton fällt,

ist γ auf $[0,\pi]$ injektiv.

Zu $x \in [-1,1]$ ex. ein $t \in [0,\pi]$

mit $\quad \cos(t) = x$

$$0 < \sin(t) = \sqrt{1-\cos^2(t)} = \sqrt{1-x^2} = y.$$



$$\dot\gamma(t) = \gamma'(t) = (-\sin t, \cos t)$$

$$\|\gamma'(t)\|_2 = \sqrt{\sin^2(t) + \cos^2(t)} = 1$$

$$\text{Schul-Sin}(t) = \frac{\text{Gegenkathete}}{\text{Hypotenuse}} = \frac{\sin t}{1}$$

$$\tan t = \frac{\sin t}{\cos t}$$

$$\mathbb{R} \setminus \{ t : \cos t = 0 \}$$

$$= \mathbb{R} \setminus \{ \tfrac{\pi}{2} + n\pi : n \in \mathbb{Z} \}$$

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = -\frac{\sin t}{\cos t} = \tan t$$

$$\tan(t+\pi) = \frac{\sin(t+\pi)}{\cos(t+\pi)} = \frac{-\sin(t)}{-\cos(t)} =$$

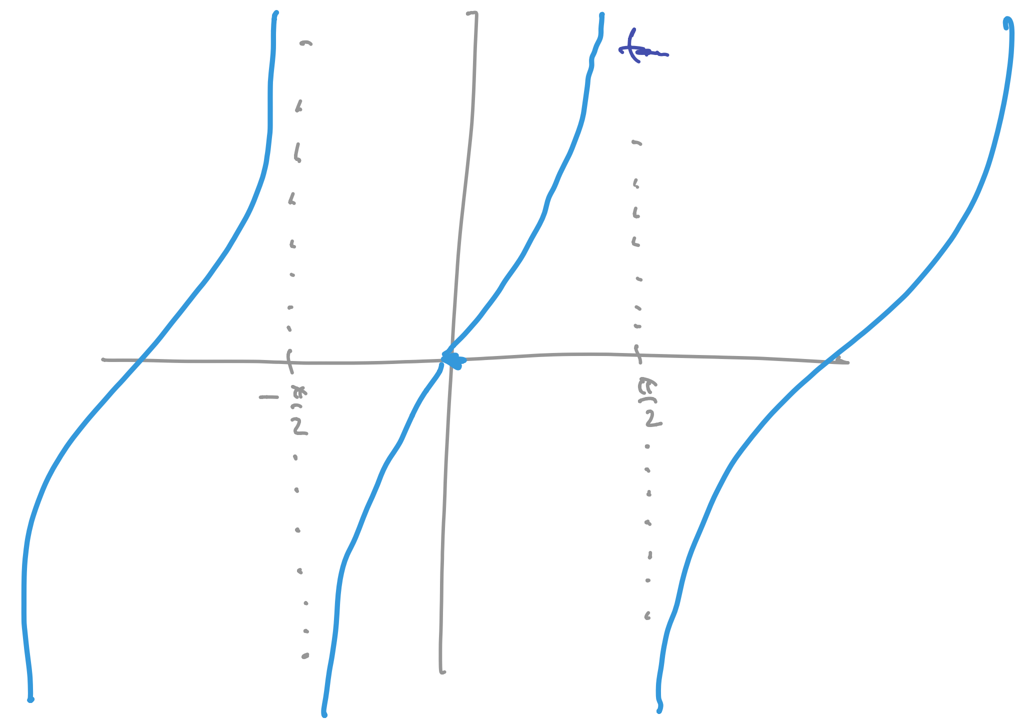
$$= \frac{\sin t}{\cos t} = \tan t.$$

$$\tan' t = \left(\frac{\sin t}{\cos t} \right)'$$

$$= \frac{s'c - sc'}{c^2} = \frac{c^2 + s^2}{c^2}$$

$$= 1 + \frac{s^2}{c^2}$$

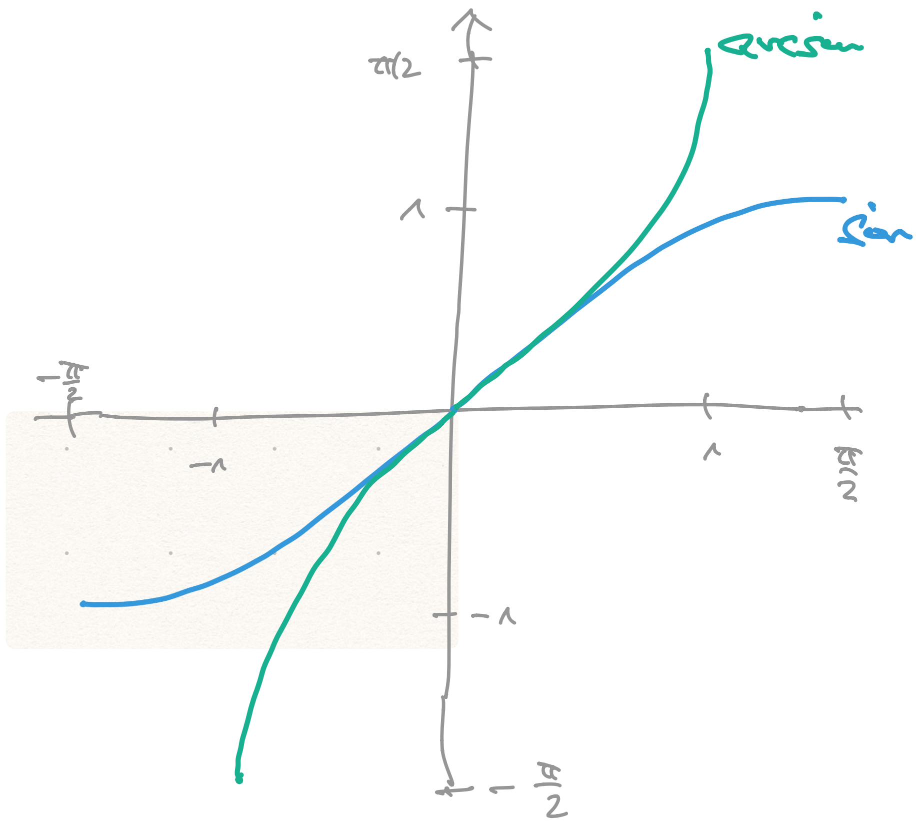
$$= 1 + \tan^2 t.$$



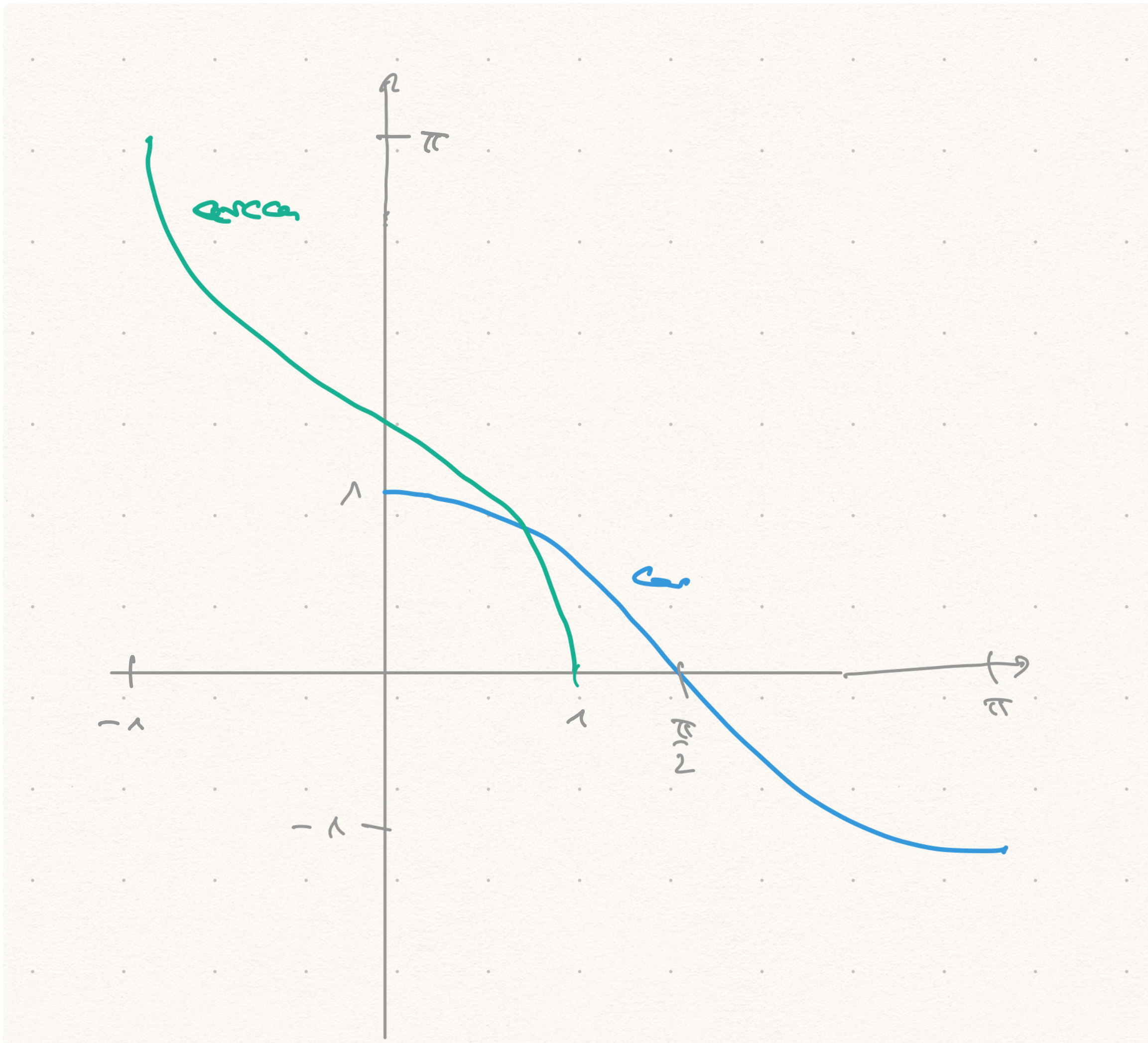
$\sin\big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$

$\cos\big|_{[0, \pi]}$

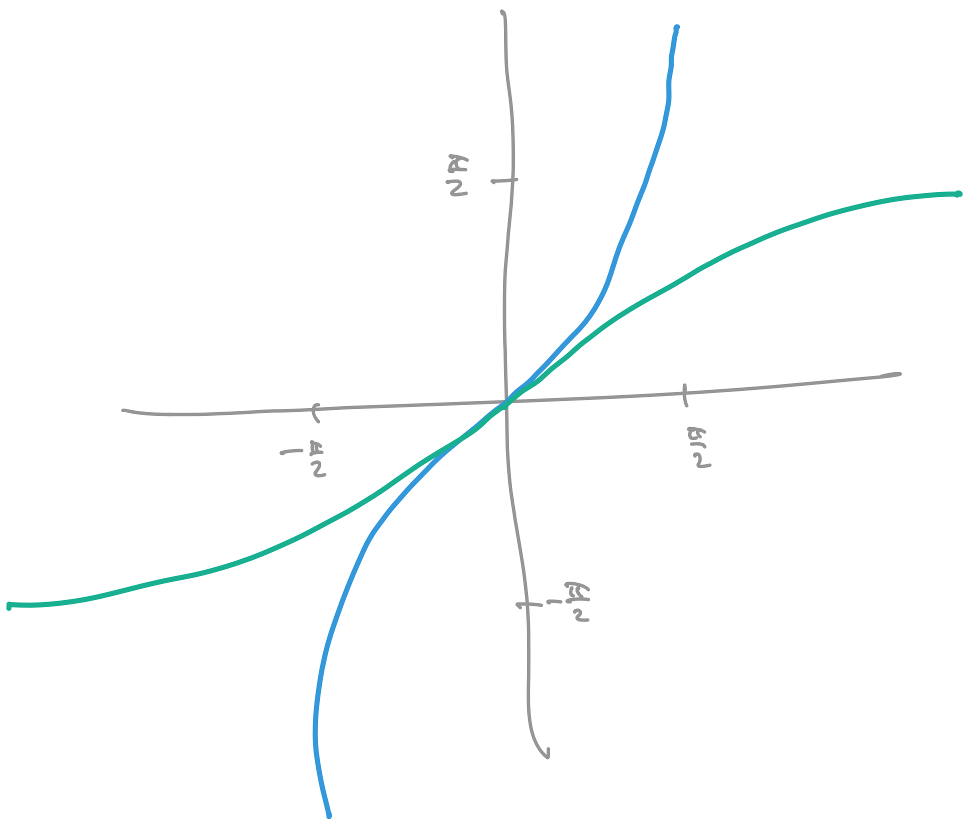
$\tan\big|_{(-\frac{\pi}{2}, \frac{\pi}{2})}$



arcsin
sin
π/2
1
−1
−π/2



π
1
-1
-1
1
π/2
π
arccos
cos



$\frac{\pi}{2}$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$-\frac{\pi}{2}$

Ableitung:

$$\arcsin'(t) = \frac{1}{\sin'(s)}\Big|_{s=\arcsin t}$$

$$= \frac{1}{\underbrace{\cos(s)}_{>0}}\Big|_{s=\ldots}$$

$$= \frac{1}{+\sqrt{1-\sin^2(s)}}\Big|_{s=\arcsin t}$$

$$= \frac{1}{\sqrt{1-t^2}}, \qquad |t|<1.$$

$$\arctan' t = \frac{1}{\tan' s}\Big|_{s=\arctan t}$$

$$= \frac{1}{1+\tan^2 s}\Big|_{s=\ldots}$$

$$= \frac{1}{1+t^2}, \qquad t\in\mathbb{R}.$$